Hale School

Mathematics Specialist

Test 3 --- Term 2 2016

Vectors in 3D

/ 41

Name: **ANSWERS**

Instructions:

- **CAS calculators are allowed**
- **External notes are not allowed**
- **Duration of test: 50 minutes**
- **Show your working clearly**
- **Use the method specified (if any) in the question to show your working (Otherwise, no marks awarded)**
- **This test contributes to 7% of the year (school) mark**

Question 1 (7 marks: 1, 2, 4)

A sphere has its centre at $C(1, 1, 0)$ and radius 3.

(a) State the Cartesian equation of the sphere.

$$(x-1)^2 + (y-1)^2 + z^2 = 3^2$$

States the answer ✓

Consider a diameter with end points at $P(3, 3, 1)$ and $Q(a, b, c)$ on the sphere.

(b) Determine the values of a, b and c.

$$\mathbf{QC} = \mathbf{CP}$$

$$\Rightarrow \begin{pmatrix} 1-a \\ 1-b \\ -c \end{pmatrix} = \begin{pmatrix} 3-1 \\ 3-1 \\ 1 \end{pmatrix}$$

$$\Rightarrow a = -1,\ b = -1,\ c = -1$$

Uses **QC** = **CP** ✓

States the answers ✓

Let $X(x, y, z)$ be any point (except P and Q) on the sphere.

(c) Prove that **PX** is perpendicular to **QX**.

$$\mathbf{PX} = \begin{pmatrix} x-3 \\ y-3 \\ z-1 \end{pmatrix} \quad \text{and} \quad \mathbf{QX} = \begin{pmatrix} x+1 \\ y+1 \\ z+1 \end{pmatrix}$$

$$\mathbf{PX} \cdot \mathbf{QX} = \begin{pmatrix} x-3 \\ y-3 \\ z-1 \end{pmatrix} \cdot \begin{pmatrix} x+1 \\ y+1 \\ z+1 \end{pmatrix}$$

$$= x^2 - 2x - 3 + y^2 - 2y - 3 + z^2 - 1$$

$$= (x-1)^2 + (y-1)^2 + z^2 - 9$$

$$= 0 \qquad (\because \text{ X is on the sphere})$$

$$\therefore \quad \mathbf{PX} \perp \mathbf{QX}$$

Determines **PX** and **QX** ✓

Determines their dot product ✓

Uses X on the sphere to simplify the equation ✓

Shows that the dot product is zero ✓

Question 2 (3 marks: 1, 2)

The Cartesian equation of a plane π is given by $x - 2y + z = 3$ and the vector equation of a line ℓ is given by $\vec{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

(a) State a normal vector to π.

$$\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

States the answer ✓

(b) State the vector equation of the plane Λ which contains the line ℓ and parallel to π.

$$\Lambda: \quad \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{So} \quad \Lambda: \quad \mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -3$$

Uses the normal vector of π as the normal vector of Λ ✓

States the answer ✓

Question 3 (6 marks: 2, 4)

An object, A, with initial position vector $\mathbf{r}_A(0) = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ metres is moving with velocity $\mathbf{v}_A = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ m/s.

A second object, B, with initial position vector $\mathbf{r}_B(0) = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ metres is moving with velocity $\mathbf{v}_B = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ m/s.

(a) Find the positions of A and B at time t.

$$\mathbf{r}_A(t) = \begin{pmatrix} 2+3t \\ 3-t \\ -1+4t \end{pmatrix} \quad \text{and} \quad \mathbf{r}_B(t) = \begin{pmatrix} 5+at \\ 3+bt \\ 2+ct \end{pmatrix}$$

Uses **r**(t) = **r**(0) + t**v** ✓

States the answers ✓

(b) If $|\mathbf{v}_B| = \sqrt{14}$ m/s and A and B collide, find the time(s) of collision.

Collision $\Rightarrow$ $\mathbf{r}_A(t) = \mathbf{r}_B(t)$ for some $t > 0$

$$\Rightarrow \begin{pmatrix} 2+3t \\ 3-t \\ -1+4t \end{pmatrix} = \begin{pmatrix} 5+at \\ 3+bt \\ 2+ct \end{pmatrix}$$

$$\Rightarrow \quad a = 3 - \frac{3}{t}, \quad b = -1 \quad \text{and} \quad c = 4 - \frac{3}{t}$$

$$\Rightarrow \quad \left(3 - \frac{3}{t}\right)^2 + 1 + \left(4 - \frac{3}{t}\right)^2 = 14$$

$$\Rightarrow \quad t = \frac{1}{2}\ \text{s} \quad \text{or} \quad 3\ \text{s}$$

Equates $\mathbf{r}_A(t)$ and $\mathbf{r}_B(t)$ for collision ✓

Expresses a, b and c in terms of t ✓

Uses the given speed to set up the equation relating a, b and c ✓

Solves for t correctly ✓

Question 4 (7 marks: 4, 2, 1)

Given the equations of two planes: $\pi_1: x - y + z = 1$ and $\pi_2: x - z = 4$.

(a) Find the vector equation of the line which π_1 and π_2 intersect.

$\pi_1: x - y + z = 1 \quad \ldots\ldots \quad (1)$

$\pi_2: x \quad - z = 4 \quad \ldots\ldots \quad (2)$

Let $z = \lambda$

(2) $\Rightarrow x = 4 + \lambda$

(1) $\Rightarrow 4 + \lambda - y + \lambda = 1$

$\Rightarrow y = 3 + 2\lambda$

So $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4+\lambda \\ 3+2\lambda \\ \lambda \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Uses a parameter to solve π_1 and π_2 ✓

Expresses the other variables in terms of the parameter ✓

Writes **r** as <x, y, z> ✓

Writes **r** in terms of the parameter ✓

A third plane is given by $\pi_3: 4x - 3y + 2z = d$ where d is an unknown.

(b) (i) Determine the value of d if the three equations $\begin{cases} x - y + z = 1 \\ x - z = 4 \\ 4x - 3y + 2z = d \end{cases}$ have many solutions.

$3 \times \pi_1 + \pi_2: \quad 4x - 3y + 2z = 7$

many solutions $\Rightarrow d = 7$

Or sub **r** into π_3 and solve for d.

Adds $3 \times \pi_1$ to π_2 ✓

States the value of d ✓

(ii) Given the solutions in (b) (i), provide a geometric interpretation of the three planes in (b) (i).

The three planes meet at a common line.

Gives the correct interpretation ✓

Question 5 (4 marks: 2, 1, 1)

Given three vectors $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 0 \\ -9 \\ h \end{pmatrix}$ where h is an unknown.

(a) Given that $\mathbf{b}\times\mathbf{c} = 18\,\mathbf{i} - 36\,\mathbf{j} - 18\,\mathbf{k}$, determine the value of h.

$\mathbf{b}\times\mathbf{c} = 18\mathbf{i} - 36\mathbf{j} - 18\mathbf{k}$

$(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})\times(0\mathbf{i} - 9\mathbf{j} + h\,\mathbf{k}) = 18\mathbf{i} - 36\mathbf{j} - 18\mathbf{k}$

$\mathbf{i}:\ -h + 36 = 18$

$\therefore\quad h = 18$

Considers the cross product to give **i** or **j** component. ✓

Sets up and solve the equation ✓

(b) Evaluate $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

$= (\mathbf{i} + 4\mathbf{j} - 7\mathbf{k})\cdot(18\mathbf{i} - 36\mathbf{j} - 18\mathbf{k})$

$= 18 - 144 + 126$

$= 0$

States the correct answer ✓

(c) Give a geometric interpretation regarding the three vectors of your answer in part (b).

The three vectors lie in the same plane.

Gives the correct interpretation ✓

Question 6 (7 marks: 2, 2, 3)

A particle moves along a path described by the vector function $\mathbf{r}(t) = 2\sin(\frac{t}{2})\mathbf{i} + 3\cos(\frac{t}{2})\mathbf{j}$ for $0 \le t \le 2\pi$.

(a) Determine the Cartesian equation of the path.

$$\mathbf{r}(t) = 2\sin(\frac{t}{2})\mathbf{i} + 3\cos(\frac{t}{2})\mathbf{j}$$

$$\Rightarrow \quad x = 2\sin(\frac{t}{2}) \quad \text{and} \quad y = 3\cos(\frac{t}{2})$$

$$\Rightarrow \quad (\frac{x}{2})^2 + (\frac{y}{3})^2 = 1$$

Identifies x and y components	✓
States the answer	✓

(b) Determine the velocity function.

$$\mathbf{r}(t) = 2\sin(\frac{t}{2})\mathbf{i} + 3\cos(\frac{t}{2})\mathbf{j}$$

$$\Rightarrow \quad \mathbf{v}(t) = \cos(\frac{t}{2})\mathbf{i} - \frac{3}{2}\sin(\frac{t}{2})\mathbf{j}$$

Differentiates $\mathbf{r}(t)$	✓
States the answer	✓

(c) Determine the maximum speed.

$$\mathbf{v}(t) = \cos(\frac{t}{2})\mathbf{i} - \frac{3}{2}\sin(\frac{t}{2})\mathbf{j}$$

$$\Rightarrow \quad \text{speed}^2 = \cos^2(\frac{t}{2}) + \frac{9}{4}\sin^2(\frac{t}{2})$$

$$\Rightarrow \quad = 1 + \frac{5}{4}\sin^2(\frac{t}{2})$$

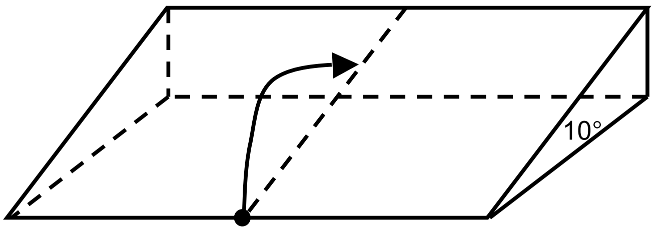
$$\text{So} \quad \text{speed}_{max} \Leftrightarrow \sin^2(\frac{t}{2}) = 1$$

$$\text{And} \quad \text{speed}_{max} = \frac{3}{2}$$

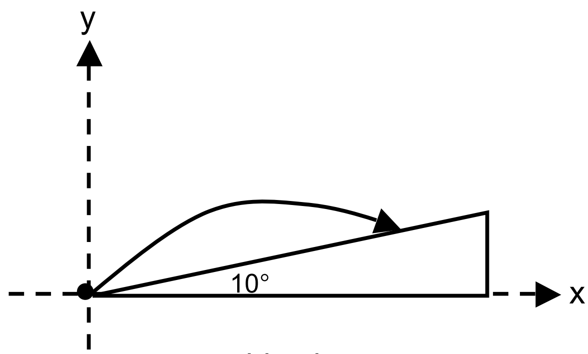
Writes speed in terms of t	✓
States the condition on $\sin^2(t/2)$ for max speed	✓
States the answer	✓

Question 7 (7 marks: 3, 2, 2)

A particle, at the bottom of an inclined plane, is projected up the plane along a line of the greatest slope as shown below.



3D view



side view

The initial velocity of the particle is 40 m/s making an angle 20° with the **plane**.

The particle experiences an acceleration of 10 m/s^2 downwards throughout its motion. Ignore air resistance and use vector calculus in answering the following questions.

(a) Determine the velocity vector of the particle at time t.

$$\mathbf{a} = -10\,\mathbf{j}$$
$$\mathbf{v}(t) = -\int 10\,\mathbf{j}\,dt$$
$$= -10\,t\,\mathbf{j} + \mathbf{c}_1$$
$$\mathbf{v}(0) = \mathbf{c}_1 = 40\cos(30°)\mathbf{i} + 40\sin(30°)\mathbf{j}$$
$$\mathbf{v}(t) = 20\sqrt{3}\,\mathbf{i} + (20 - 10\,t)\,\mathbf{j}$$

Writes **a** = -10 **j** ✓

Integrates **a** to get **v** ✓

Uses **v**(0) to find $\mathbf{c}_1$ ✓

(b) Determine the position vector of the particle at time t.

$$\mathbf{v}(t) = 20\sqrt{3}\,\mathbf{i} + (20 - 10\,t)\,\mathbf{j}$$
$$\mathbf{r}(t) = \int [20\sqrt{3}\,\mathbf{i} + (20 - 10\,t)\,\mathbf{j}]\,dt$$
$$= 20\sqrt{3}\,t\,\mathbf{i} + (20\,t - 5\,t^2)\,\mathbf{j} + \mathbf{c}_2$$
$$\mathbf{r}(t) = \mathbf{c}_2 = 0\,\mathbf{i} + 0\,\mathbf{j}$$
$$\therefore\quad \mathbf{r}(t) = 20\sqrt{3}\,t\,\mathbf{i} + (20\,t - 5\,t^2)\,\mathbf{j}$$

Integrates **v** to get **r** ✓

Uses **r**(0) to find $\mathbf{c}_2$ ✓

(c) Determine the time when the particle hits the plane.

hits the plane $\Rightarrow \frac{y}{x} = \tan 10°$

$\Rightarrow \frac{20t - 5t^2}{20\sqrt{3}t} = \tan 10°$

$\Rightarrow t = 2.78$ s

Sets y/x = tan 10°	✓
States the answer	✓